Drift of spiral waves in the complex Ginzburg-Landau equation due to media inhomogeneities

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We test the asymptotical theory of dynamics of spiral waves by applying it to inhomogeneity-induced drift of the spiral waves in the Complex Ginzburg-Landau equation for two different types of weak media inhomogeneities and demonstrate good quantitative agreement with numerical simulations for both.

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INTRODUCTION

Spiral waves are a specific form of self-organization, first reported in the Belousov-Zhabotinsky reaction medium [1] and observed in a variety of physical, chemical, and biological systems [2]. Spatial inhomogeneity of the system usually leads to the drift of the spirals; this is seen in experiments [3] and reproduced in numerical simulations [4]. Attempts to explain or predict the direction and the velocity of the drift have been made, based on various phenomenological arguments applicable to narrow classes of autowave media [4,5]. The method of the *response functions* (RF's) [6,7] describes dynamics of spiral waves in terms of Aristotelean dynamics, so that the velocities of the drift in space and time are proportional to the forces caused by the perturbation. The theory claims to provide a *universal* and *quantitatively accurate* tool for describing drift of spiral waves due to a small perturbation, including a small and/or smooth inhomogeneity. RF's, which are the critical eigenfunctions of the adjoint linearized operator, were first introduced to autowave media to describe the dynamics of twisted and bent three-dimensional scroll waves, and were assumed to be asymptotically periodic in space like the spiral waves themselves. That led to a necessity of artificial regularization procedures [8]. Later a hypothesis about the essential localization of the RF's in the vicinity of the spiral wave core was proposed [6] and then used to describe dynamics of the spiral and scroll waves [7,9]. A variety of experimental phenomenology, showing the insensitivity of spiral waves to distant events, supported the hypothesis, but the mathematical peculiarity of the idea, which presumed qualitatively different behavior of eigenfunctions of a linear operator and its adjoint, resulted in a natural skepticism. Although the existence of the RF's (as they are solutions to overdetermined problems) in general is an open question, they have been found numerically for some particular models [10–12] and shown to be localized for these cases. Explicit knowledge of localized RF's for all sets of parameters in the complex Ginzburg-Landau equation (CGLE) for which stable spiral wave solutions exist, allows us to test the predictive ability of the theory.

In this paper, we demonstrate this predictive ability for the inhomogeneity-induced drift of spiral waves in CGLE, by showing good quantitative agreement of the predictions with the results of direct numerical simulations, for two particular types of inhomogeneity, the gradient of the linear frequency, and the gradient of the nonlinear dispersion coefficient.

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THE GENERAL THEORY

In this section we briefly recapitulate the general theory of spiral wave drift proposed in Ref. [7].

Usually spiral waves are modeled by reaction-diffusion systems of partial differential equations

$$\partial_t u = f(u) + D\nabla^2 u + \epsilon h(u,x,t),$$

$$u,f \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l\times l}, l \geqslant 2.$$

In unperturbed media, at $\epsilon=0$, we assume that a solution in the form of a steadily rotating wave exists,

$$u = U(\mathbf{r},t) = U(\rho(\mathbf{r}), \theta(\mathbf{r}) + \omega t). \tag{1}$$

This rotating wave will be a spiral wave, if $u(\rho,\phi) \approx \tilde{U}(\rho/\Lambda - \phi/2\pi)$ as $\rho \to \infty$, for a $\tilde{U}(\xi)$:mod(1), $\tilde{U} \not\equiv$ const. Then equiphase lines at large ρ are close to Archimedean spirals with pitch Λ.

If a spiral wave solution (1) exists, then

$$\tilde{u} = U(\rho(\mathbf{r}-\mathbf{R}), \theta(\mathbf{r}-\mathbf{R}) + \Theta),$$

where $\Theta = \omega t - \Phi$, is another solution for any constant $\mathbf{R}$, Φ. This is a spiral wave shifted in space by $\mathbf{R}$ and rotated by Φ, or equivalently, shifted in time by Φ/ω. Thus, the unperturbed reaction-diffusion system in $\mathbb{R}^2$ has a three-dimensional manifold of spiral wave solutions, parametrized by two-dimensional vector $\mathbf{R}$ and phase Θ. Physical observability implies that this manifold is reasonably stable as a whole.

A perturbation $\epsilon h \neq 0$ could be a slight inhomogeneity of the medium or an explicit time-dependent external forcing. Typical effects of the perturbation on the stable invariant manifold of spiral waves are (i) a displacement of the manifold and (ii) a perturbed dynamics along this manifold. The latter is a slow change of previously constant parameters $\mathbf{R}$ and Φ, i.e., spatial and temporal drift of the spiral wave (the temporal drift is the shift of the rotation frequency),

$$\partial_t \Theta = \omega + \epsilon H_0(\mathbf{R},\Theta), \quad \partial_t \mathbf{R} = \epsilon \mathbf{H}_1(\mathbf{R},\Theta), \tag{2}$$

where the second equation can also be written as $\partial_t R = \epsilon H_1(\mathbf{R},\Theta)$, where $R \equiv \mathbf{R}_x + i\mathbf{R}_y$ and $H_1 \equiv \mathbf{H}_{1,x} + i\mathbf{H}_{1,y}$.

The velocities ϵH_0, ϵH_1 of these drifts, in the first approximation, are linear functionals of the perturbation. Both H_0 and H_1, after sliding averaging over the spiral wave rotation period, can be expressed as

$$\overline{H}_n(t) = e^{in\Phi} \oint_{t-\pi/\omega}^{t+\pi/\omega} \frac{\omega\, d\tau}{2\pi} \int_{\mathbb{R}^2} d^2\mathbf{r}\ \ e^{-in\omega\tau}$$

$$\times \langle W_n(\rho(\mathbf{r}-\mathbf{R}), \theta(\mathbf{r}-\mathbf{R}) + \omega\tau - \Phi), h\rangle, \qquad (3)$$

where $h = h(U(\mathbf{r},\tau), \mathbf{r}, \tau)$, $\mathbf{R} = \mathbf{R}(t)$, $\Phi = \Phi(t)$, and W_n called *response functions*, $n = 0, \pm 1$ are the critical eigenfunctions,

$$L^+ W_n = -i\omega n W_n, \quad n = 0, \pm 1,$$

of the adjoint linearized operator

$$L^+ = D\boldsymbol{\nabla}^2 + \omega\partial_\theta + \left(\frac{\partial f}{\partial u}\right)^T\Bigg|_{u=U(\mathbf{r})},$$

chosen to be biorthogonal to the Goldstone modes,

$$V_0 = -\partial_\theta U(\rho(\mathbf{r}), \theta(\mathbf{r}))|_{t=0},$$

$$V_{\pm 1} = -\tfrac{1}{2} e^{\mp i\theta}(\partial_\rho \mp i\rho^{-1}\partial_\theta) U(\rho(\mathbf{r}), \theta(\mathbf{r}))|_{t=0},$$

which are the critical eigenfunctions of the linearized operator,

$$L = D\nabla^2 - \omega\partial_\theta + \left(\frac{\partial f}{\partial u}\right)\Bigg|_{u=U(\mathbf{r})}.$$

RESULTS

We apply the asymptotic theory of the spiral wave dynamics [7] to the perturbed CGLE, which is a two-component reaction-diffusion system conveniently presented in the complex form

$$\partial_t u = u - (1-\mathcal{I}\alpha)u|u|^2 + (1+\mathcal{I}\beta)\boldsymbol{\nabla}^2 u + \epsilon h, \qquad (4)$$

where $u \in \mathbb{C}$, α, $\beta \in \mathbb{R}$, and $\mathcal{I}$ is the imaginary unit. The above asymptotic theory is applied to real-valued systems of equations. Equation (4) can be rewritten as a two-component real system, e.g. for the real and imaginary parts of u; then $\mathcal{I}$ would be represented by the matrix

$$\mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For calculations, it was convenient to keep treating Eq. (4) as a complex equation. The complexification of the linearized theory then leads to an algebra with two imaginary units, $\mathcal{I}$ from Eq. (4), and i of the linearized theory, with $i^2 = \mathcal{I}^2 = -1$ and $i\mathcal{I} = \mathcal{I}i$. It has divisors of zero, e.g., $(i+\mathcal{I})(i-\mathcal{I}) = 0$. See also Ref. [11] for details of realification of Eq. (4).

The steadily rotating spiral wave solutions to this equation have been studied by Hagan [13] and have the form

$$U(\mathbf{r},t) = e^{\mathcal{I}(\theta+\omega t)} P(\rho),$$

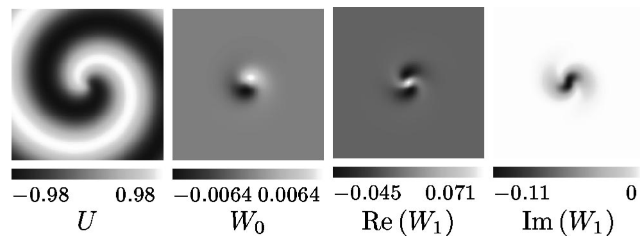

FIG. 1. Spiral wave and response functions for $\alpha = 0.1$ and $\beta = 0.6$.

where $P(\rho) = a(\rho)e^{\mathcal{I}\psi(\rho)} \in \mathbb{C}$, and ω solve a nonlinear eigenvalue problem

$$(1+\mathcal{I}\beta)\left(P'' + \frac{1}{\rho}P' - \frac{1}{\rho^2}P\right)$$

$$+ [1 - \mathcal{I}\omega - (1-\mathcal{I}\alpha)|P|^2]P = 0,$$

$$P(\rho \to 0) \propto \rho,$$

$$P(\rho \to \infty) \approx \sqrt{1-k^2}\, \exp[\mathcal{I}k\rho + o(\rho)][1 + o(1)],$$

$k = k(\alpha, \beta)$ is the asymptotic wave number and

$$\omega = \alpha - \alpha k^2 - \beta k^2.$$

The response functions have the form [11]

$$W_n = e^{(\mathcal{I} - in)\theta} Q_n(\rho), \qquad (5)$$

where Q_n, $n = 0, 1$ are solutions to linear problems

$$(1-\mathcal{I}\beta)\left\{Q_n'' + \frac{1}{\rho}Q_n' + \frac{(\mathcal{I}-in)^2}{\rho^2}Q_n\right\}$$

$$+ \{1 + \mathcal{I}\omega - a^2[2(1+\mathcal{I}\alpha) + (1-\mathcal{I}\alpha)e^{2\mathcal{I}\psi}\mathcal{C}]\}Q_n = 0, \quad (6)$$

$$|Q_n(\rho \to 0)| < \infty, \quad Q_n(\rho \to \infty) \to 0. \qquad (7)$$

Here $\mathcal{C}$ is the operator of $\mathcal{I}$ conjugation; W_1 and Q_1 are bicomplex-valued functions, each having four components.

Solutions to Eqs. (6) and (7) were found numerically in the form

$$Q_0(\rho) = [A(\rho) + \mathcal{I}B(\rho)]e^{\mathcal{I}\psi(\rho)},$$

$$Q_1(\rho) = [C(\rho) + \mathcal{I}D(\rho) + iE(\rho) + i\mathcal{I}F(\rho)]e^{\mathcal{I}\psi(\rho)}, \quad (8)$$

where the functions C, D, E, and F were tabulated.

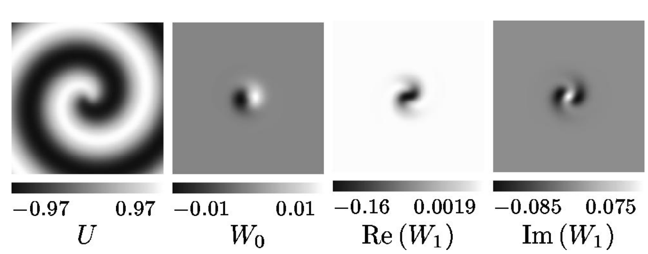

FIG. 2. Spiral wave and response functions for $\alpha = 0$ and $\beta = -1$.

Let us now consider the perturbation

$$h=\mathcal{I}x|u|^2u, \tag{9}$$

which corresponds to a linear inhomogeneity in the parameter α: $\tilde{\alpha}(\mathbf{r})=\alpha+\epsilon x$. Substitution of Q_1 from Eqs. (8) and (9) into Eqs. (5), (3), and (2), with account of the normalization $\langle W_j(a),V_k(a)\rangle=\delta_{j,k}$, gives the following expression for the spiral wave drift velocity due to the inhomogeneity of the medium (9):

$$\partial_t R=\epsilon H_1=\frac{\epsilon\int_0^\infty[D-iF]a^3\rho^2\ \ d\rho}{\int_0^\infty\{aF-\rho(a'C+a\psi'D)+i[aD+\rho(a'E+a\psi'F)]\}\ \ d\rho}, \tag{10}$$

and zero for the frequency correction H_0, as the perturbation h in Eq. (9) is an even function and the response function W_0 in Eq. (5) is an odd function.

Thus, if we know Hagan's solution, a,ψ and the components of the response functions, C, D, E, and F, Eq. (10) gives a theoretical prediction for the spiral wave drift velocity due to the inhomogeneity (9).

The spiral wave solution $U(\mathbf{r})$ and response functions $W_0(\mathbf{r})$ and $W_1(\mathbf{r})$ for $\alpha=0.1$ and $\beta=0.6$, are shown in Fig. 1.

The integral (10) calculated for these functions predicts the normalized velocities $\partial_t\mathbf{R}_x/\epsilon=\mathrm{Re}(H_1)\approx-1.958$ and $\partial_t\mathbf{R}_y/\epsilon=\mathrm{Im}(H_1)\approx-29.137$. This has been calculated at the spatial discretization step $h_\rho=0.01$. Comparison with velocities calculated at larger h_ρ has shown that these values are accurate within 0.1%.

To check the theoretical prediction, we numerically solved the CGLE (4) with the perturbation (9) of amplitude $\epsilon=0.0001$. We used the first-order fully explicit time-stepping scheme with five-point approximation of the Laplacian. The computational grid was of spatial size 128×128, with discretization steps h_t from 0.005 to 0.4 and h_x from 0.25 to 0.5. Initial conditions were specified using Hagan's solution.

The dynamics of the phase of the spiral for this perturbation is not interesting (see above) and has not been considered here. The trajectory of the center of the spiral was defined as the intersection of the null isolines of $\mathcal{I}$-real and $\mathcal{I}$-imaginary parts of u. This trajectory was used to measure the velocity of the drift: after a short transient the trajectories become straight lines that were fitted by linear functions to find the drift velocities. We measured the components of the normalized drift velocity, $\partial_t\mathbf{R}_x/\epsilon$ and $\partial_t\mathbf{R}_y/\epsilon$ in numeric simulations, and their behavior as $h_x\to0$ and $h_t\to0$. The crucial parameter limiting the convergence to the theoretical value was the spatial discretization step of the numerical simulation. At the smallest steps used in simulations, $h_t=0.005$ and $h_x=0.25$, the components of the normalized drift velocity were $\partial_t R_x/\epsilon=-1.923$ and $\partial_t R_y/\epsilon=-29.09$, so the difference from the theoretical value was less than 2%. Thus, predictions of the asymptotic theory were in very good quantitative agreement with the results of direct numerical simulations, up to the precision achievable by these simulations.

We have also verified the theory on the perturbation

$$h=\mathcal{I}xu, \tag{11}$$

to compare the prediction with the recently published numerical results for this case [14]. For the perturbation (11), the theory gives the following expression for the velocity of the spiral wave drift:

$$\partial_t R=\epsilon H_1=\frac{\epsilon\int_0^\infty[D-iF]a\rho^2\ \ d\rho}{\int_0^\infty\{aF-\rho(a'C+a\psi'D)+i[aD+\rho(a'E+a\psi'F)]\}\ \ d\rho},$$

and zero for the frequency correction H_0, again.

Coefficients $m_{\omega,\|}$ and $m_{\omega,\perp}$ of Ref. [14] coincide to $\mathbf{H}_{1,x}$ and $\mathbf{H}_{1,y}$ from Eq. (2) if inhomogeneity is in the form (11). We have calculated Hagan's solution, a,ψ and the components of the response functions, C, D, E, and F for α in the interval $[-1,0]$, which corresponds to $\alpha\in[0,1]$ in Ref. [14] due to different choice of the sign, at fixed $\beta=-1$. This interval crosses the Eckhaus instability line at $\alpha\approx-0.4$ [12] so that its beginning is before the Eckhaus instability line while the end is quite beyond it. The spiral waves and the components of the RF's for these two ending points of the parameter interval are shown in Figs. 2 and 3.

The resulting velocities are shown in Fig. 4. They are indistinguishable from results obtained by another method [14] (which were published for $0\leqslant\alpha\leqslant0.8$). More than that, the calculations using RF predict that $\mathbf{H}_{1,x}$ changes the sign at $\alpha\approx-0.87$. This is an interesting qualitative prediction that could be checked by numerical simulations; however, in

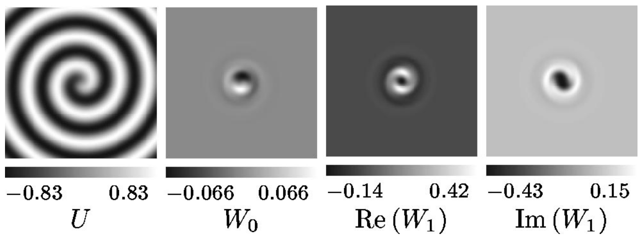

FIG. 3. Spiral wave and response functions for $\alpha = -1$ and $\beta = -1$.

this particular region of parameters it is not easy because of the Eckhaus instability of the spiral.

CONCLUSIONS

The method of the response functions allows us to predict, *quantitatively*, the velocity of the spiral wave drift due to various weak media inhomogeneities in the CGLE without any restrictions on the type of an inhomogeneity. These predictions are obtained by a computationally much less expensive way than direct numerical simulations: for the CGLE, instead of solving systems of partial differential equations in two spatial dimensions+time, only the response functions for each particular set of parameters need to be found (which is a solution of a 1D boundary value problem). After that the spiral wave dynamics following any slight perturbation can be predicted with a good quantitative precision by just the calculation of the integral (3).

The quantitative advantage of the method of the response functions over direct simulations may sometimes give new qualitative results. An example is the change of the sign of the longitudinal velocity coefficient $\mathbf{H}_{1,x}$ as $\beta = -1$ and $\alpha \approx -0.87$ and inhomogeneity is of the form (11). This phenomenon is difficult to see by the direct simulations because of the Eckhaus instability of the spiral. The advantage of the method of the response functions allows us to learn this point

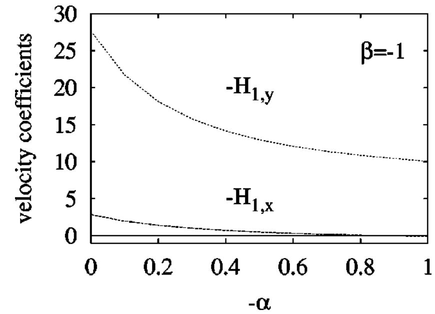

FIG. 4. Velocity coefficients as functions of α.

and to continue the line $\mathbf{H}_{1,x}$ in the (α,β) plane to a physically observable region.

Bearing in mind that the existence of the response functions, in a general case, is still an open question, the good quantitative predictive ability of the theory allows us to speak about a new phenomenon: *qualitatively* different behavior of eigenfunctions of a linear operator and its adjoint one (see Figs. 1, 2, and 3), as the hypothesis about local sensitivity of spiral waves [6] and the asymptotic theory of spiral wave dynamics [7] now has been confirmed directly and quantitatively for the CGLE.

The important fact is that due to the localization of the sensitivity of spiral waves to perturbations, including resonance drift of spiral waves [15] and drift due to weak spatial inhomogeneities, the dynamics of spiral waves may be considered as dynamics of effectively localized ''particles,'' despite their nonlocal appearance, unlike solitons.

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[1] A. T. Winfree, Science **175**, 634 (1972).
[2] M. A. Allessie, F. I. M. Bonk, and F. J. G. Schopman, Circ. Res. **32**, 54 (1973); F. Alcantara and M. Monk, J. Gen. Microbiol. **85**, 321 (1974); N. A. Gorelova and J. Bures, J. Neurobiol. **14**, 353 (1983); S. Jakubith *et al.*, Phys. Rev. Lett. **65**, 3013 (1990); P. Coullet and F. Plaza, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 1173 (1994); D. Yu, W. Lu, and R. G. Harrison, J. Opt. Soc. Am. B **1**, 25 (1999).
[3] V. G. Fast and A. M. Pertsov, Biofizika **35**, 478 (1990) (in Russian); M. Markus, Z. Nagy-Ungvarai, and B. Hess, Science **257**, 225 (1992).
[4] R. Hramov, A. Rudenko, A. Panfilov, and V. Krinskij, Stud. Biophys. **102**, 69 (1984); A. Pertsov and E. Ermakova, Biofizika **33**, 338 (1988) (in Russian); G. Ivanitsky, V. Krinsky, A. Panfilov, and M. Tsyganov, *ibid.* **34**, 296 (1989) (in Russian).
[5] V. Krinsky, Problemy Kibernetiki **2**, 59 (1968) (in Russian); A. Panfilov and A. Holden, Phys. Lett. A **147**, 463 (1990); I. Aranson and M. Rabinovich, J. Phys. A **23**, 299 (1990); A. Mikhailov, V. Davydov, and V. Zykov, Physica D **70**, 1 (1994); Y. Elkin and V. Biktashev, J. Biol. Phys. **25**, 129 (1999); V. Hakim and A. Karma, Phys. Rev. E **60**, 5073 (1999).
[6] V. N. Biktashev, Ph.D. thesis, Moscow Institute of Physics and Technology, 1989.
[7] V. N. Biktashev and A. V. Holden, Chaos, Solitons Fractals **5**, 575 (1995).
[8] J. P. Keener, Physica D **31**, 269 (1988).
[9] V. N. Biktashev, A. V. Holden, and H. Zhang, Philos. Trans. R. Soc. London, Ser. A **347**, 611 (1994).
[10] E. Hamm, Ph.D. thesis, Université de Nice–Sophia Antipolice/ Institut Non Lineair de Nice, 1997 (in French).
[11] I. V. Biktasheva, Y. E. Elkin, and V. N. Biktashev, Phys. Rev. E **57**, 2656 (1998).
[12] I. V. Biktasheva and V. N. Biktashev, J. Nonlin. Math. Phys. (to be published).
[13] P. S. Hagan, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **42**, 762 (1982).
[14] M. Hendrey, E. Ott, and T. M. Antonsen, Phys. Rev. Lett. **82**, 859 (1999).
[15] I. V. Biktasheva, Y. E. Elkin, and V. N. Biktashev, J. Biol. Phys. **25**, 115 (1999).